

String Origins

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Abstract

It is shown that the string concept results naturally from considerations of gravitation. This paper describes a derivation of linearized general relativity based upon the hypotheses of special covariance and the existence of a gravitational potential. The gravitational field possesses gauge invariance given by a second-order covariant derivative defining an associated differential geometry. The concepts of parallelism and parallel transport lead to string-like constructions.

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1 Introduction

Strings were originally introduced as a means of describing hadronic interactions [6], [7], [10]. After the quantum theory was found to contain the graviton it became a candidate for unification of the known forces [8], [9]. The discovery of the anomaly cancellation [3] gave further impetus to the unification programme leading to the search for realistic models [4], [5]. The character of this paradigm shift from the zero-dimensional particle, which had been the hypothesized elementary constituent of matter from the time of the early Greeks until the latter part of the twentieth century, to the one-dimensional string has appeared ad hoc, lacking theoretical continuation from prior held knowledge of relativity and quantum fields. It is the purpose of this paper to show that the string concept emanates naturally from considerations of gravitation.

This paper presents a derivation of linearized general relativity based upon two hypotheses: special covariance and the existence of a gravitational potential. The approach is in close analogy with electrodynamics, which provides the motivation behind most of the concepts to be introduced. The gravitational potential defines a gravitational field tensor which possesses gauge invariance given by second-order covariant differentiation, in contradistinction to first-order Yang-Mills theories. The differential geometry associated to the gauge transformations yields the graviton, a perturbation of the metric from a flat spacetime background. Moreover, the arbitrariness of the geometric structure corresponds precisely to the gauge freedom of the graviton, identifying the theory with perturbative general relativity [1], [2]. The concept of parallel transport leads to string-like constructions.

In the following section we consider how gravitation at low energy may be represented covariantly by a field on a flat Lorentzian spacetime. This is accomplished by a three-index tensor. Then in Section 3 it is shown that the gravitational field is determined by the potential

by means of Lorentz covariance and symmetry requirements. The gravitational potential, $A_{\mu\nu}$, is shown to be a symmetric, two-index tensor field. The next section justifies the hypothesized existence of the gravitational potential. The wave equation is applied to the potential of a single point mass, analogously to the Liénard-Wiechert potential in classical electrodynamics. The solution is used to determine the gravitational field tensor and in the static, low-energy limit one recovers the Newtonian theory. The final section explores the geometry associated to second-order covariant differentiation and its connections to string theory.

The gravitational field tensor possesses gauge invariance given by a second-order derivative on a line bundle. The concept of second-order covariant differentiation is defined in analogy to the first-order case appearing in Yang-Mills theory and requires the involvement of an auxiliary first-order covariant derivative defined by a vector field A_μ . The related geometry lives on the manifold of strings embedded into the spacetime \mathcal{M} , which describes the underlying space for string second quantization [11]. It is shown that \mathcal{M} must be flat and the second-order covariant derivative defines a fluctuation from this flat background. Thus the spacetime manifolds are restricted to those that admit a flat Lorentzian metric. The notion of a flat background from which gravitation propagates is therefore a mathematical consequence of the theory.

The second-order covariant derivative reveals the composite structure of the graviton in terms of the gravitational potential $A_{\mu\nu}$ and the vector field A_μ . The vector field defines the gauge transformations of the graviton and consequently the hypothesized gravitational potential is seen to represent the graviton in an arbitrary choice of gauge. The related geometry is based upon a worldsheet in spacetime rather than a worldline, hence its association with string-like objects. The symmetric tensor part of the second-order covariant derivative replicates

the bosonic NS-NS fields and couples to the string in a similar manner. Parallel translation is carried out over a surface rather than along a line and operates on the direct sum of two copies of the string Hilbert space of states.

2 The Gravitational Field Tensor

On a flat Lorentzian manifold, a covariant tensor representing the gravitational field is constructed. This tensor field defines an affine connection whose geodesics determine the path followed by a particle.

Let Q be a particle with rest mass M_0 and 4-position $q(t) = (t, \mathbf{q}(t))$ on a flat Lorentzian manifold \mathcal{M} with coordinates $x = (t, \mathbf{x})$. The metric is given by $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ and $c = 1$ in what follows. Retarded time $t_0 = t_0(x)$ is defined by $t - t_0 = |\mathbf{x} - \mathbf{q}(t_0)|$; gravity is taken to propagate at the speed of light. The gravitational field of Q will be constructed from the following covariant expressions depending upon at most second-order derivatives of q^μ :

$C^\mu(x)$, the lightlike vector field defined by $C^\mu(x) = x^\mu - q^\mu(t_0)$,

$U^\mu(x) = \frac{dq^\mu}{d\tau}(t_0)$, where τ is proper time as measured by Q ,

$\dot{U}^\mu(x) = \frac{d^2q^\mu}{d\tau^2}(t_0)$,

$X = C^\mu U_\mu$,

$Y = C^\mu \dot{U}_\mu$, and

$Z = \dot{U}^\mu \dot{U}_\mu$.

Define the three-index Lorentz-covariant tensor field

$$H = -GM_0 X^{-3} C \wedge U \otimes U, \quad (1)$$

where the wedge product is given by $C \wedge U = C \otimes U - U \otimes C$ and G is the gravitational constant. It is convenient to consider a symmetrized form of H :

$$F^\mu{}_{\nu\lambda} = \frac{1}{2}(H^\mu{}_{\nu\lambda} + H^\mu{}_{\lambda\nu}). \quad (2)$$

An affine connection $\Gamma = \Gamma^\mu{}_{\nu\lambda}$ on the manifold \mathcal{M} may be defined by

$$\Gamma = \nabla_0 - F, \quad (3)$$

where ∇_0 denotes the flat Lorentzian connection.

Consider the geodesic equations

$$\frac{d^2 x^\mu}{ds^2} + \Gamma_{\nu\lambda}^\mu \frac{dx^\nu}{ds} \frac{dx^\lambda}{ds} = 0. \quad (4)$$

Since $F_{(\mu\nu\lambda)} = 0$, every timelike solution of (4) can be parametrized so that s denotes proper time for x^μ with respect to the Lorentzian metric. If Q has uniform velocity then in a Lorentz frame where it is at rest at the origin, the geodesic equations (4) are equivalent to the Newtonian force law describing the motion $x^\mu = x^\mu(t)$ of a ponderomotive particle under the influence of the gravitational field of Q :

$$\mathbf{F} = -GM_0 m \mathbf{x} / r^3, \quad (5)$$

where $r = \sqrt{\mathbf{x} \cdot \mathbf{x}}$ and m is relativistic mass. Therefore, at low energy $F_{\mu\nu\lambda}$ defines the gravitational field of a nonaccelerating point mass in covariant form.

3 The Gravitational Potential

The electrodynamic field tensor is determined locally by a 1-form potential. Analogously, we hypothesize that to the gravitational field tensor F is associated a potential A . The latter must be a two-index tensor field and the relationship given by the ansatz

$$F_{\mu\nu\lambda} = a\partial_\mu A_{\nu\lambda} + b\partial_\mu A_{\lambda\nu} + c\partial_\nu A_{\lambda\mu} + d\partial_\nu A_{\mu\lambda} + e\partial_\lambda A_{\mu\nu} + f\partial_\lambda A_{\nu\mu}, \quad (6)$$

for constants a, b, c, d, e and f . Symmetry considerations will determine all constants appearing in (6) up to an overall constant multiple which may be absorbed into A . First we show that A must be a symmetric tensor field.

In order to ensure that the equations of motion (4) do not involve derivatives of q^μ higher than second order, $A_{\mu\nu}$ can depend upon at most first-order derivatives. That is, $A_{\mu\nu}$ must be expressible in terms of X , U^μ and C^μ only:

$$A_{\mu\nu} = a(X)U_\mu U_\nu + b(X)U_\mu C_\nu + c(X)C_\mu U_\nu + d(X)C_\mu C_\nu, \quad (7)$$

where a, b, c and d are smooth functions. In the static case where Q resides at the origin, rotational symmetry requires that $b = c = d = 0$. It follows that $A_{\mu\nu}$ has the form

$$A_{\mu\nu} = a(X)U_\mu U_\nu. \quad (8)$$

In particular, $A_{\mu\nu}$ is symmetric.

The symmetry relations

$$F_{\mu\nu\lambda} = F_{\mu\lambda\nu}, \quad (9)$$

$$F_{\mu\nu\lambda} + F_{\nu\lambda\mu} + F_{\lambda\mu\nu} = 0, \quad \text{and} \quad (10)$$

$$A_{\mu\nu} = A_{\nu\mu} \quad (11)$$

yield

$$F_{\mu\nu\lambda} = \partial_\mu A_{\nu\lambda} - \frac{1}{2}(\partial_\nu A_{\mu\lambda} + \partial_\lambda A_{\mu\nu}) \quad (12)$$

and

$$H_{\mu\nu\lambda} = \partial_\mu A_{\nu\lambda} - \partial_\nu A_{\mu\lambda}. \quad (13)$$

4 Consistency

The theory is now defined by a symmetric potential $A_{\mu\nu}$ which determines the gravitational field by means of Equation (12). In electrodynamics, the Liénard-Wiechert potential describing the field of a single point charge solves the wave equation in empty space. Analogously, we shall solve the wave equation for the potential $A_{\mu\nu}$ of the particle Q given by the form (8) and determine the resultant gravitational field tensor.

Remark: The results that follow actually hold for a much more general ansatz:

$$\begin{aligned} A_{\mu\nu} &= aC_\mu C_\nu + bC_\mu U_\nu + cC_\mu \dot{U}_\nu \\ &+ dU_\mu C_\nu + eU_\mu U_\nu + fU_\mu \dot{U}_\nu \\ &+ g\dot{U}_\mu C_\nu + h\dot{U}_\mu U_\nu + k\dot{U}_\mu \dot{U}_\nu, \end{aligned} \quad (14)$$

for arbitrary smooth functions a, b, c, d, e, f, g, h and k of the scalars X, Y and Z , with the requirement that the partials $\partial_\mu A_{\nu\lambda}$ contain no third-order derivatives of q^μ .

The equation $\square A_{\mu\nu} = 0$ in empty space has the solution

$$A_{\mu\nu} = aX^{-1}U_\mu U_\nu, \quad (15)$$

where a is a constant. This may be found with the aid of the following derivatives:

$$\begin{aligned} t_{0,\mu} &= \gamma X^{-1}C_\mu, \quad \text{where } \gamma = [1 - (\frac{d\mathbf{q}}{dt}(t_0))^2]^{-1/2}, \\ C_{\mu,\nu} &= \eta_{\mu\nu} - X^{-1}U_\mu C_\nu, \\ U_{\mu,\nu} &= X^{-1}\dot{U}_\mu C_\nu, \\ \dot{U}_{\mu,\nu} &= X^{-1}\ddot{U}_\mu C_\nu, \quad \text{where } \ddot{U}_\mu(x) = \frac{d^3 q_\mu}{d\tau^3}(t_0), \\ X_{,\nu} &= U_\nu + X^{-1}(Y - 1)C_\nu, \\ Y_{,\nu} &= \dot{U}_\nu + X^{-1}C^\mu \ddot{U}_\mu C_\nu, \quad \text{and} \\ Z_{,\nu} &= 2X^{-1}\dot{U}^\mu \ddot{U}_\mu C_\nu. \end{aligned} \quad (16)$$

There are two observations to be made.

- (i) In the case where Q is at rest at the origin, $A_{\mu\nu} = \frac{a}{r}\delta_{0\mu}\delta_{0\nu}$. Therefore the A_{00} component corresponds to the Newtonian potential when a is defined to be $a = -GM_0$.
- (ii) The gravitational field H determined from (15) is

$$H = -GM_0 X^{-3}[C \wedge (U + X\dot{U}) \otimes U + C \wedge U \otimes (-YU + X\dot{U})]. \quad (17)$$

Restricting to the case where Q has uniform motion gives

$$H = -GM_0 X^{-3}C \wedge U \otimes U. \quad (18)$$

Remarkably, this is the covariant gravitational field tensor given in (1); this indicates there is something correct about the approach here taken. Hence the wave equation applied to the potential in empty space leads to Newtonian gravitation in the static limit. This provides justification for the hypothesized existence of the gravitational potential. In fact, we shall see in Section 5 that the potential represents the graviton.

5 Gauge Invariance and Geometry

The gravitational field tensor F does not uniquely determine the potential A . Rather, A has the gauge freedom

$$A_{\mu\nu} \rightarrow A_{\mu\nu} + \partial_\mu \partial_\nu \Lambda, \quad (19)$$

where Λ is an arbitrary smooth function. This differs from gauge theories based upon differential forms in that it involves a second-order derivative. As gauge theory is intimately tied to differential geometry the associated geometric constructions will be considered next. This will lead to the identification of the gravitational potential $A_{\mu\nu}$ with the graviton and the interpretation of the gravitational field theory described in Sections 2-4 as linearized general relativity. Furthermore, string-like entities will appear as the natural generalization, in the second-order formalism, of geometric objects in conventional (first-order) differential geometry. The approach we will follow shall be to define concepts in analogy to the first-order case. We begin with the notion of second-order covariant differentiation.

Let \mathcal{M} denote a Lorentzian manifold. A second-order covariant derivative on a vector bundle $E \rightarrow \mathcal{M}$ is a linear map

$$D^{(2)} : \Omega^0(E) \rightarrow \mathcal{T}^2(\mathcal{M}) \otimes_{\Omega^0(\mathcal{M})} \Omega^0(E), \quad (20)$$

which satisfies a Leibniz rule involving an auxiliary first-order covariant derivative $D^{(1)}$ on $E \oplus T\mathcal{M}$:

$$D^{(2)}(f\xi) = D^{(1)}(df) \otimes \xi + S(df \otimes D^{(1)}\xi) + fD^{(2)}\xi. \quad (21)$$

$\mathcal{T}^k(\mathcal{M}) = \Gamma^\infty(T_k^0(\mathcal{M}))$ and $S : \mathcal{T}^2(\mathcal{M}) \otimes_{\Omega^0(\mathcal{M})} \Omega^0(E) \rightarrow \mathcal{T}^2(\mathcal{M}) \otimes_{\Omega^0(\mathcal{M})} \Omega^0(E)$ is the linear map defined by

$$S(\theta^1 \otimes \theta^2 \otimes Y) = (\theta^1 \otimes \theta^2 + \theta^2 \otimes \theta^1) \otimes Y, \quad (22)$$

for $Y \in \Omega^0(E)$ and $\theta^1, \theta^2 \in \mathcal{T}^1(\mathcal{M})$. Equation (21) is chosen so as to mimic the Leibniz rule for the square D^2 of a (first-order) covariant

derivative. The auxillary first-order covariant derivative is a necessary ingredient since second-order differentiation applied to a product of a function and a vector field will include terms differentiated to first order only. $D^{(1)}$ is defined to be the direct sum of two connections

$$D^{(1)} = D_E^{(1)} \oplus D_{\mathcal{M}}^{(1)}, \quad (23)$$

where $D_E^{(1)}$ denotes a connection on E and $D_{\mathcal{M}}^{(1)}$ is the Levi-Civita connection on \mathcal{M} . $D_E^{(1)}$ is arbitrary and shall be seen to correspond to gauge freedom of the graviton. Latin indices i, j, k will indicate coordinates with respect to a local frame ξ_i for E and Greek indices μ, ν, λ will represent the spacetime coordinates for \mathcal{M} :

$$D_E^{(1)}(\xi_j) = (1/G)A_{\mu j}^i dx^\mu \otimes \xi_i \text{ and} \quad (24)$$

$$D_{\mathcal{M}}^{(1)}(dx^\mu) = -A_{\nu\lambda}^\mu dx^\nu \otimes dx^\lambda. \quad (25)$$

$D^{(2)}$ determines a tensor field $K = K^i_{j\mu\nu}$ by

$$D^{(2)} = (D^{(1)})^2 + (1/G)K. \quad (26)$$

The notion of parallel transport on E for $D^{(2)}$ is sought in a manner analogous to that for a (first-order) covariant derivative along a curve $X^\mu : [0, 1] \rightarrow \mathcal{M}$:

$$[f^i_{,\mu} + (1/G)A_{\mu j}^i f^j]|_{X^\rho(t)} \dot{X}^\mu = 0. \quad (27)$$

To couple the spacetime variables X^μ to the second-order covariant derivative we separate components from $D^{(2)}$ that are tensors with respect to the local spacetime symmetries of \mathcal{M} and with respect to the gauge transformations associated to E . Furthermore, the coupling of a component to X^μ should be a Lie algebra-valued scalar with respect to reparameterization. The salient feature as concerns the first-order case is that (27) may be expressed as a set of ordinary differential equations for $u^i(t) = f^i(X^\rho(t))$:

$$\dot{u}^i + (1/G)A_{\mu j}^i \dot{X}^\mu u^j = 0. \quad (28)$$

The requirement of corresponding differential equations in the second-order case will place restrictions upon the spacetime manifold \mathcal{M} .

Consider first the $(D^{(1)})^2$ part of $D^{(2)}$. $(D^{(1)})^2$ decomposes into symmetric and anti-symmetric parts:

$$(D^{(1)})^2 = (D^{(1)})^2_{sym} + \frac{1}{2}R, \quad (29)$$

where $R = R^i_{j\mu\nu}$ is the curvature for $D_E^{(1)}$. Due to anti-symmetry in the μ and ν indices, $R^i_{j\mu\nu}$ cannot couple to the vector field \dot{X}^μ , defined along a worldline, to produce a non-zero Lie algebra-valued scalar. Therefore the coupling of X^μ to the anti-symmetric part must be of the form

$$\frac{1}{2\sqrt{|h|}}\epsilon^{ab}R^i_{j\mu\nu}X^\mu{}_{,a}X^\nu{}_{,b}, \quad (30)$$

where $X^\mu : \Sigma \rightarrow \mathcal{M}$ maps a surface Σ into \mathcal{M} , ϵ^{ab} is the anti-symmetric pseudo-tensor and h denotes the determinant of a metric h_{ab} on Σ . Thus the extra index produced by second-order differentiation demands a geometry based upon a worldsheet in place of a worldline; hence the appearance of strings.

The coupling for the symmetric part of $(D^{(1)})^2(\xi)$ is

$$h^{ab}(D^{(1)})^2(\xi)_{\mu\nu}X^\mu{}_{,a}X^\nu{}_{,b}. \quad (31)$$

In coordinates with $u^i(\sigma^1, \sigma^2) = f^i(X^\rho(\sigma^1, \sigma^2))$, Equation (31) contains two terms that involve f^i :

$$f^i{}_{,\mu}X^\mu{}_{,ab}h^{ab} \quad \text{and} \quad (32)$$

$$f^i{}_{,\mu}A^\mu_{\nu\lambda}X^\nu{}_{,a}X^\lambda{}_{,b}h^{ab}. \quad (33)$$

In order for (31) to describe partial differential equations for u^i , in suitable coordinates, these two terms must vanish:

- (i) X^μ satisfies the equation $X^\mu{}_{,ab}h^{ab} = 0$, and
- (ii) there exist coordinates on \mathcal{M} such that $A^\mu_{\nu\lambda} = 0$. That is, \mathcal{M} is a flat Lorentzian manifold. This places a topological restriction upon the possible spacetime manifolds. In particular, the total Chern class of \mathcal{M} must be zero.

Next, consider the K term. It suffices to restrict the discussion to a line bundle E ; henceforth $A_{\mu j}^i$ shall be denoted by A_μ and $R_{j\mu\nu}^i$ by $R_{\mu\nu}$. With respect to the flat background $\eta_{\mu\nu}$ on \mathcal{M} , K decomposes into three irreducible components:

$$K_{\mu\nu} = \chi_{\mu\nu} + B_{\mu\nu} + \phi\eta_{\mu\nu}, \quad (34)$$

where $\chi_{\mu\nu}$ is symmetric and traceless, $B_{\mu\nu}$ is skew-symmetric and ϕ is a scalar. $\chi_{\mu\nu}$ may be interpreted as the graviton field, describing a perturbation $g_{\mu\nu} = \eta_{\mu\nu} + \chi_{\mu\nu}$ from the flat metric $\eta_{\mu\nu}$ and $(1/G)B_{\mu\nu}$ as a type of curvature term. Let $R^{(2)}$ be the scalar curvature for h_{ab} . When the curvatures vanish and $g_{\mu\nu} = \eta_{\mu\nu}$ then parallel transport should reduce to $h^{ab}u_{,ab} = 0$, in suitable coordinates. This suggests that the scalar ϕ be multiplied with $R^{(2)}$ in the equation defining parallel transport:

$$P_1 + P_2 + P_3 + P_4 = 0, \quad (35)$$

where

$$\begin{aligned} P_1 &= h^{ab}[u_{,ab} + \frac{2}{G}A_\mu X^\mu{}_{,a} u_{,b} + (\frac{1}{G}A_{\mu,\nu} + \frac{1}{G^2}A_\mu A_\nu)X^\mu{}_{,a} X^\nu{}_{,b} u], \\ P_2 &= \frac{1}{2\sqrt{|h|}}\epsilon^{ab}R_{\mu\nu}(X^\rho)X^\mu{}_{,a} X^\nu{}_{,b} u, \\ P_3 &= \frac{1}{G}[h^{ab}\chi_{\mu\nu}(X^\rho) + \frac{1}{\sqrt{|h|}}\epsilon^{ab}B_{\mu\nu}(X^\rho)]X^\mu{}_{,a} X^\nu{}_{,b} u, \text{ and} \\ P_4 &= R^{(2)}\phi(X^\rho)u. \end{aligned} \quad (36)$$

The K field replicates the bosonic NS-NS fields $\chi_{\mu\nu}$, $B_{\mu\nu}$ and ϕ , and couples to the string X^μ in a similar manner.

The vector field A_μ transforms as $A_\mu \rightarrow A_\mu + \partial_\mu\Lambda$ with respect to a gauge transformation on the line bundle E , whereas the potential $A_{\mu\nu}$ must transform as $A_{\mu\nu} \rightarrow A_{\mu\nu} + \partial_\mu\partial_\nu\Lambda$. The appearance of the term $\frac{1}{2}(\partial_\mu A_\nu + \partial_\nu A_\mu)$ within the symmetric part of $D^{(2)}$ motivates the identification

$$A_{\mu\nu} = \chi_{\mu\nu} + \frac{1}{2}(\partial_\mu A_\nu + \partial_\nu A_\mu). \quad (37)$$

Therefore, the gauge transformation on E is given by

$$A_{\mu\nu} \rightarrow A_{\mu\nu} + \partial_\mu \partial_\nu \Lambda, \quad (38)$$

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda, \quad (39)$$

$$f \rightarrow \exp(-\Lambda/G)f, \quad \text{and} \quad (40)$$

$$K_{\mu\nu} \rightarrow K_{\mu\nu}. \quad (41)$$

Furthermore, it is seen from (37) that $\frac{1}{2}(\partial_\mu A_\nu + \partial_\nu A_\mu)$ acts as the gauge transformation for the graviton:

$$\chi_{\mu\nu} \rightarrow \chi_{\mu\nu} + \frac{1}{2}(\partial_\mu A_\nu + \partial_\nu A_\mu). \quad (42)$$

The arbitrary nature of the auxiliary first-order covariant derivative in the definition of second-order covariant differentiation naturally describes the gauge freedom of a graviton. We thus have two separate groups of gauge transformations.

Since a traceless gauge of type (38), $\eta^{\mu\nu} A_{\mu\nu} = 0$, may be chosen, $A_{\mu\nu}$ represents the graviton. Therefore, the preceding developments have led to linearized general relativity.

Lastly, consider the complexification of E . For h_{ab} with metric normal form $(+1, -1)$, Equation (35) is hyperbolic and determined by initial conditions

$$\begin{aligned} u(0, \sigma) & \quad 0 \leq \sigma \leq \pi, \quad \text{and} \\ u_{,\tau}(0, \sigma) & \quad 0 \leq \sigma \leq \pi. \end{aligned} \quad (43)$$

A continuous linear functional $\Psi = \Psi[f]$ of the space of fields f on $[0, \pi]$ can be represented by a continuous complex function g through the inner product

$$\Psi[f] = \langle g, f \rangle = \int_0^\pi g^* f. \quad (44)$$

These functionals are identified in the Schrödinger representation with the Hilbert space $\mathcal{H}(\tau)$ of string states associated to the string $X^\mu(\tau, \sigma)$ at fixed τ . Conversely, the functions $u(\tau, \sigma)$ and $u_{,\tau}(\tau, \sigma)$ each define a state in the Hilbert space. Parallel transport along a surface therefore maps $\mathcal{H}(\tau_1) \oplus \mathcal{H}(\tau_1)$ into $\mathcal{H}(\tau_2) \oplus \mathcal{H}(\tau_2)$.

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